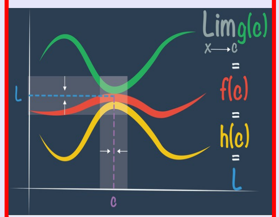
Calculus I

Lecture 3



Feb 19-8:47 AM

Class QZ 3

Box Your Final Answer

1) Evaluate $\lim_{x \to 3} \frac{x^2-5x+6}{x^2-9} = \frac{3^2-5(3)+6}{3^2-9} = \frac{0}{0}$ I.F.

$= \lim_{x \to 3} \frac{(x-2)\cancel{(x-3)}}{(x+3)\cancel{(x-3)}} = \lim_{x \to 3} \frac{x-2}{x+3} = \frac{3-2}{3+3} = \boxed{\frac{1}{6}}$

2) Evaluate $\lim_{x \to 4} \frac{2x-8}{\sqrt{x}-2} = \frac{2(4)-8}{\sqrt{4}-2} = \frac{0}{0}$ I.F.

$= \lim_{x \to 4} \frac{2(x-4)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)} = \lim_{x \to 4} \frac{2\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{x-4}} = \lim_{x \to 4} 2(\sqrt{x}+2)$

$= 2(\sqrt{4}+2) = \boxed{8}$

Jun 18-7:24 AM

Evaluate $\lim_{x \to 0} \dfrac{\frac{1}{x+2} + \frac{1}{x-2}}{x} = \dfrac{\frac{1}{0+2} + \frac{1}{0-2}}{0} = \dfrac{\frac{1}{2} - \frac{1}{2}}{0} = \dfrac{0}{0}$ I.F.

LCD $(x+2)(x-2)$

$$= \lim_{x \to 0} \frac{\cancel{(x+2)}(x-2) \cdot \frac{1}{\cancel{x+2}} + (x+2)\cancel{(x-2)} \cdot \frac{1}{\cancel{x-2}}}{(x+2)(x-2) \cdot x}$$

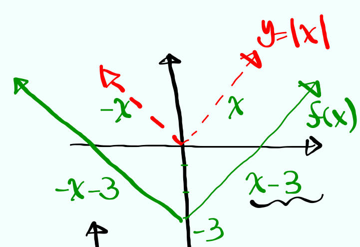
$$= \lim_{x \to 0} \frac{x \cancel{-2} + x \cancel{+2}}{(x+2)(x-2) \cdot x} = \lim_{x \to 0} \frac{2\cancel{x}}{(x+2)(x-2) \cdot \cancel{x}}$$

$$= \lim_{x \to 0} \frac{2}{(x+2)(x-2)}$$

$$= \frac{2}{(0+2)(0-2)} = \frac{2}{-4} = \boxed{\frac{-1}{2}}$$

Jun 18-8:20 AM

Given $f(x) = |x| - 3$



$$\lim_{x \to 4} f(x) = \lim_{x \to 4} (|x| - 3)$$

$$= |4| - 3 = 4 - 3 = \boxed{1}$$

$$\lim_{x \to 4} (x-3) = 4-3 = 1$$

$$\lim_{x \to -5} f(x) = \lim_{x \to -5} (|x| - 3) = |-5| - 3 = 5 - 3 = \boxed{2}$$

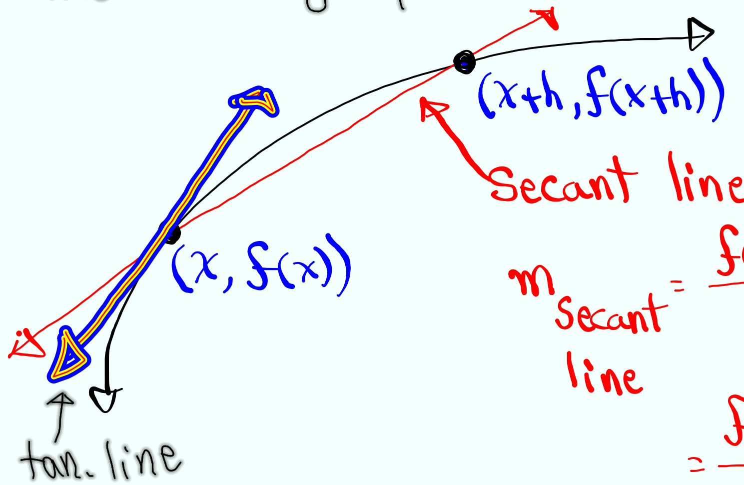
$$\lim_{x \to -5} (-x-3) = -(-5) - 3 = 5 - 3 = \boxed{2}$$

$$\boxed{|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}}$$

Important

Jun 18-8:30 AM

Consider the graph of $y=f(x)$ below



$$m_{\text{secant line}} = \frac{f(x+h)-f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}$$

Difference Quotient

$$\boxed{m_{\text{tan. line}} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

Jun 18-8:35 AM

Find slope of the tan. line to the graph of $f(x)=x^2+4x+2$

$$m=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2+4(x+h)+2-x^2-4x-2}{h}$$

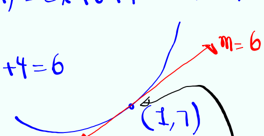
$$=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2\cancel{+4x}+4h\cancel{+2}\cancel{-x^2}\cancel{-4x}\cancel{-2}}{h}$$

$$=\lim_{h\to 0}\frac{2xh+h^2+4h}{h}=\lim_{h\to 0}\frac{\cancel{h}(2x+h+4)}{\cancel{h}}$$

$$=\lim_{h\to 0}(2x+h+4)=2x+0+4=2x+4$$

at $x=1 \rightarrow m=2(1)+4=6$

$(1,7)$



$y-y_1=m(x-x_1)$

$y-7=6(x-1)$

$\boxed{y=6x+1}$

eqn of tan. line

Jun 18-8:41 AM

Find slope of the tan. line to the graph of $f(x) = x^3 - 4x$. $\rightarrow f(-1) = (-1)^3 - 4(-1)$
$= -1 + 4$
$= 3$

$$m_{\text{tan. line}} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - 4(x+h) - x^3 + 4x}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 \cancel{-4x} - 4h \cancel{-x^3} \cancel{+4x}}{h}$$

$$\cancel{h}(3x^2 + 3xh + h^2 - 4)$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 4h}{\cancel{h}} = \lim_{h\to 0} (3x^2 + 3xh + h^2 - 4)$$

$$= \boxed{3x^2 - 4}$$

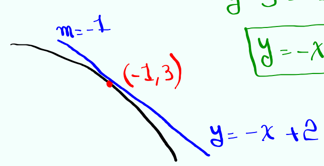
at $x = -1 \rightarrow m = 3(-1)^2 - 4 = \boxed{-1}$

$(-1, 3)$

$y - y_1 = m(x - x_1)$

$y - 3 = -1(x - (-1))$

$\boxed{y = -x + 2}$



Jun 18-8:50 AM

Find the slope of the tan. line to the graph of $f(x) = \sqrt{x}$.

$$m_{\text{tan. line}} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

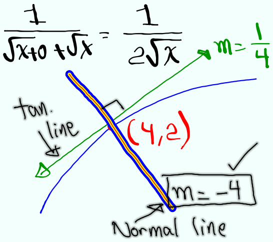
$$= \lim_{h\to 0} \frac{(\underbrace{\sqrt{x+h}}_{A} - \underbrace{\sqrt{x}}_{B})(\underbrace{\sqrt{x+h}}_{A} + \underbrace{\sqrt{x}}_{B})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{\overbrace{(\sqrt{x+h})^2}^{A^2} - \overbrace{(\sqrt{x})^2}^{B^2}}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{\cancel{x} + \cancel{h}^{\,1} \cancel{-x}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

at $x = 4 \rightarrow m = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$(4, 2)$



Jun 18-9:08 AM

Evaluate $\lim_{x \to 2} \frac{x+4}{x-2} = \frac{2+4}{2-2} = \frac{6}{0}$ undefined

If $x = 2.001$ $\quad \frac{2.001+4}{2.001-2} = \frac{6.001}{.001} = 6001$

If $x = 2.00001$ $\quad \frac{2.00001+4}{2.00001-2} = \frac{6.00001}{.00001}$

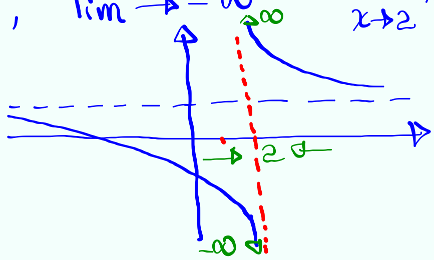
$= 600001$

As $x \to 2^+$, $\lim \to \infty$

As $x \to 2^-$, $\lim \to -\infty$

$\lim_{x \to 2} \frac{x+4}{x-2}$ D.N.E.

$y = \frac{x+4}{x-2}$



Jun 18-9:44 AM

Is $f(x) = x^3 - 5x + 1$ continuous at $x = -2$?

Method I: $f(x)$ is a polynomial function

Polynomial functions are continuous $(-\infty, \infty)$

Yes.

Method II Show $\lim_{x \to -2} f(x) \overset{\checkmark}{=} f(-2)$

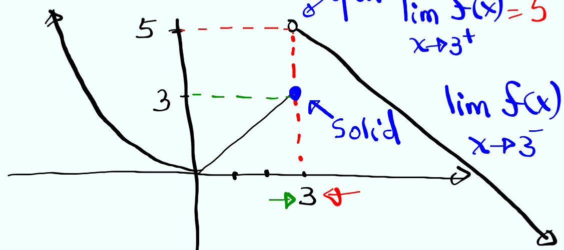
$\lim_{x \to -2} f(x) = \lim_{x \to -2} (x^3 - 5x + 1) = (-2)^3 - 5(-2) + 1$

$= -8 + 10 + 1 = \boxed{3}$

$f(-2) = (-2)^3 - 5(-2) + 1 = \boxed{3}$ **Yes**

Jun 18-9:51 AM

Consider the graph of $f(x)$ below



$\lim_{x \to 3^+} f(x) = 5$

$\lim_{x \to 3^-} f(x) = 3$

Is $f(x)$ cont. at $x=3$? NO GAP, Jump, hole at $x=3$

$\lim_{x \to 3} f(x)$ D.N.E. $f(3) = 3$

Since $\lim_{x \to 3} f(x) \neq f(3)$

Not Cont. at $x=3$

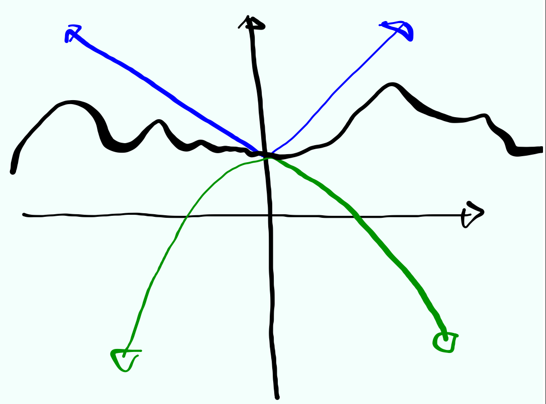
Jun 18-9:56 AM

Find $\lim_{x \to 0} f(x)$ if $1 - x^2 \leq f(x) \leq |x| + 1$.

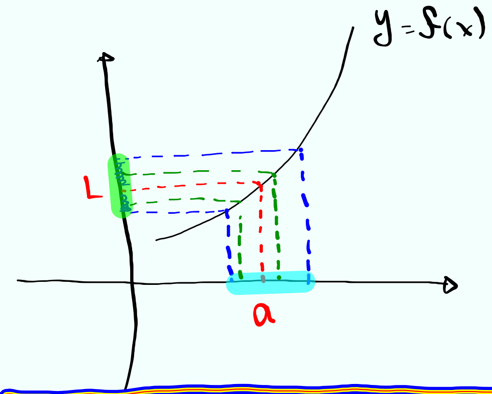
$\lim_{x \to 0} (1 - x^2) = 1$

$\lim_{x \to 0} (|x| + 1) = 1$

$\therefore \lim_{x \to 0} f(x) = 1$ by S.T.



Jun 18-10:01 AM



If $\lim_{x \to a} f(x) = L$ exists, then

Delta

for $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

Epsilon

Precise definition of limit.

Jun 18-10:08 AM

Prove $\lim_{x \to 4} (2x - 5) = 3$ using ε and δ.

$f(x) = 2x - 5$

$a = 4$

$L = 3$

1) verify the limit.

$\lim_{x \to 4} (2x - 5) = 2(4) - 5 = 8 - 5 = 3$ ✓

2) $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|2x - 5 - 3| < \varepsilon$ whenever $|x - 4| < \delta$

$|ab| = |a| \cdot |b|$

$|2x - 8| < \varepsilon$

$|2(x - 4)| < \varepsilon$

$|2|\,|x - 4| < \varepsilon$

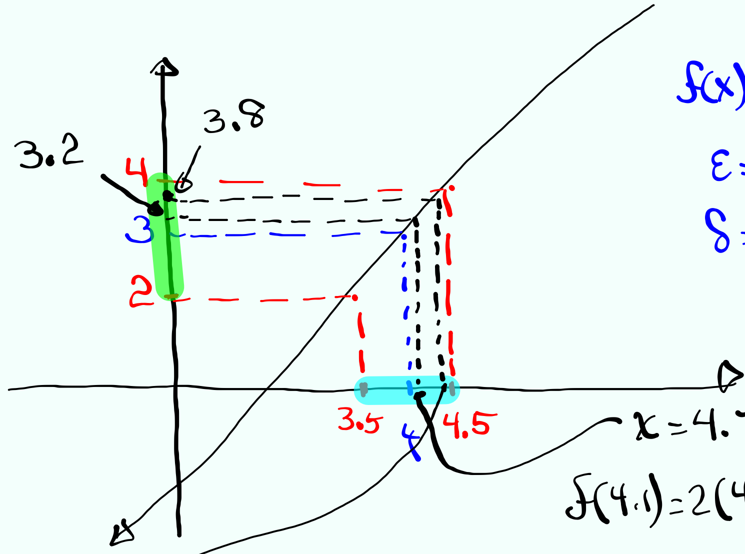
$2|x - 4| < \varepsilon \rightarrow |x - 4| < \frac{\varepsilon}{2}$

$\delta = \frac{\varepsilon}{2}$

If $\varepsilon = 1$

$\delta = \frac{1}{2}$

Jun 18-10:13 AM



$f(x)=2x-5$

$\varepsilon=1$

$\delta=\frac{1}{2}$

$x=4.1$

$f(4.1)=2(4.1)-5$

$=8.2-5$

$=3.2$

If $x=4.4$

$f(4.4)=2(4.4)-5$

$=8.8-5=\boxed{3.8}$

Jun 18-10:19 AM

Prove $\lim\limits_{x\to 4}\left(\frac{1}{2}x+3\right)=5$ using ε and δ definition.

$f(x)=\frac{1}{2}x+3$

$a=4$

$L=5$

1) verify the limit.

$\lim\limits_{x\to 4}\left(\frac{1}{2}x+3\right)=\frac{1}{2}(4)+3=2+3=5$ ✓

2) $|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$\left|\frac{1}{2}x+3-5\right|<\varepsilon$ whenever $|x-4|<\boxed{\delta}$

$\left|\frac{1}{2}x-2\right|<\varepsilon$

$\left|\frac{1}{2}(x-4)\right|<\varepsilon$

$\left|\frac{1}{2}\right||x-4|<\varepsilon$

$\frac{1}{2}|x-4|<\varepsilon$

→ Multiply by 2

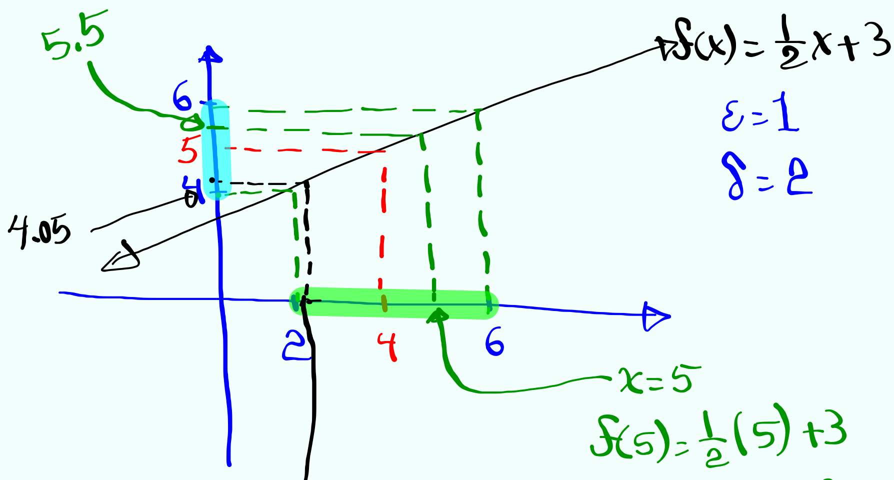
$2\cdot\frac{1}{2}|x-4|<2\varepsilon$

$|x-4|<\boxed{2\varepsilon}$

$\delta=2\varepsilon$

If $\varepsilon=1\rightarrow\delta=2$

Jun 18-10:24 AM



$x=5$

$f(5)=\frac{1}{2}(5)+3$

$=2.5+3=5.5$

$x=2.1$

$f(2.1)=\frac{1}{2}(2.1)+3=4.05$

Jun 18-10:34 AM

Prove $\lim_{x\to 2}(x^2+5x)=14$ using ε & δ def.

$f(x)=x^2+5x$

$a=2$

$L=14$

1) $\lim_{x\to 2}(x^2+5x)=2^2+5(2)$
$=4+10$
$=14$ ✓

2) $|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$|x^2+5x-14|<\varepsilon$ whenever $|x-2|<\delta$

$|(x+7)(x-2)|<\varepsilon$ whenever $|x-2|<\delta$

$\underbrace{|x+7|}_{\text{Bound}}\underbrace{|x-2|}_{\text{Keep}}<\varepsilon$

If $|x+7|<C$, then $C|x-2|<\varepsilon$, $|x-2|<\frac{\varepsilon}{C}$

It is common (for now) for $\delta\le 1$

$|x-2|<1$

$-1<x-2<1$

$1<x<3$

Add 7

$8<x+7<10$

$|x+7|<10$

$C=10$

$\delta=\min\left\{1,\frac{\varepsilon}{10}\right\}$

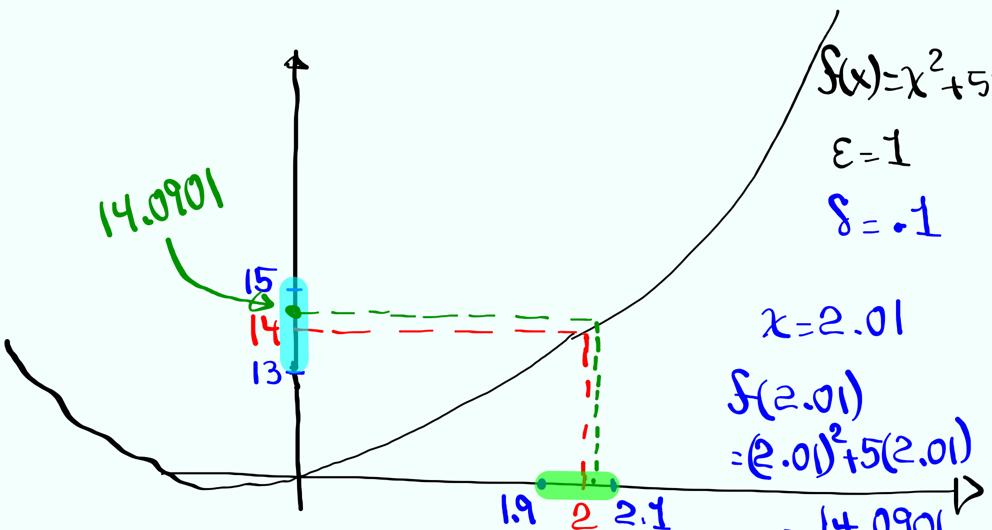
$\varepsilon=1 \to \delta=\min\{1,\frac{1}{10}\}=\frac{1}{10}$

$\varepsilon=6 \to \delta=\min\{1,\frac{6}{10}\}=.6$

$\varepsilon=1.1 \to \delta=\min\{1,\frac{1.1}{10}\}=.11$

$\varepsilon=20 \to \delta=\min\{1,\frac{20}{10}\}=1$

Jun 18-10:39 AM



$f(x) = x^2 + 5x$

$\varepsilon = 1$

$\delta = .1$

$x = 2.01$

$f(2.01)$
$= (2.01)^2 + 5(2.01)$
$= 14.0901$

Jun 18-10:53 AM